Unsteady Compressible Flow Solver

Introduction

Analysis of unsteady compressible flow problems invariably involves the resolution of shock waves internal to the computational domain. Generally, this kind problems are difficult for internal to the computational domain. Generally, these type of problems are difficult for interaction between shock waves, expansion waves and contact discontinuities as well as interaction between walls and all waves. Recently, the development of high-resolution numerical schemes, TVD schemes (Total Variation Diminishing methodology), has been a significant milestone for approaching the unsteady compressible flow. These schemes have been widely used in different numerical simulations. This part is a work that creates a new feature for STORM/CFD2000, which can handle time accuracy simulations.

Governing Equations

Hypothesis about the bulk viscosity will be used. This assumption implies that the thermodynamic pressure p is equal to one-third of the invariant sum of normal stresses. The governing equations for gas flow are as follows:

Mass conservation:

$$\frac{\partial r}{\partial t}+\frac{\partial}{\partial x_i}(rU_i)=0$$

Momentum conservation:

$$\frac{\partial rU_i}{\partial t}+\frac{\partial}{\partial x_j}(rU_iU_j)=-\frac{\partial r}{\partial x_i}+\frac{\partial}{\partial x_j}[(m+m_t)(\frac{\partial U_i}{\partial x_i}+\frac{\partial U_j}{\partial x_i})]-\frac{2}{3}(m+m_t)\frac{\partial}{\partial x_i}(\frac{\partial U_k}{\partial x_k})$$

Static enthalpy conservation:

$$\frac{\partial rh}{\partial t}+\frac{\partial}{\partial x_j}(phU_j)=\frac{dp}{dt}+\frac{\partial}{\partial x_j}[(\frac{m}{P_r}+\frac{m_t}{P_{rt}})\frac{\partial h}{\partial x_j}]+f$$

where, f is viscous energy dissipation term and defined as

$$f = [m(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) - (\frac{2}{3} m \frac{\partial U_j}{\partial x_j}) d_{ij}] \frac{\partial U_i}{\partial x_j}$$

Equation of state:

$$p = rRT$$

Where, p is the ensemble averaged density of the mixture; U_i is the ith component of the density – weighted mean velocity. In the three dimensional space, U_i has three components, $U_i = (U_1, U_2, U_3)$,. p is mean pressure; and m is the laminar viscosity.

Numerical Procedure

The method used in this study is the Pressure Implicit with Splitting Operator (PISO) algorithm first developed by Issa[1,2]. Due to the non-iterative procedure in the PISO algorithm, it has been shown to be more efficient than iterative methods, such as SIMPLE and SIMPLES. The main feature of the PISO method is to split the solution process into a series of predictor and corrector steps whereby operations on pressure are decoupled from those on velocity at each time step. The set of equations can then be solved sequentially. At each time step, the accuracy of the numerical solution is proportional to the time step size. The Charkravarthy-Osher TVD scheme[3] is used to estimate the convective flux. The control volume method was used to discretize the difference equations with variables located at each cell's center. The grid system is shown in the following figure.

NW	N	NE
W	P	E
SW	S	SE

The implicit finite difference forms of the governing equations are:

Continuity equation:

$$\frac{1}{\Delta t}(r^{n+1} - r^{n}) + \Delta_i(rU_i) = 0$$

Momentum equation:

$$\frac{1}{\Delta t}[(pU_i)^{n+1} - (pU_i)^{n}] = H(U_i^{n+1}) - \Delta_i p^{n+1} + S_{U_i}^{n+1}$$

Energy equation:

$$\frac{1}{\Delta t}[ph)^{n+1} - (ph)^{n}] = G(h^{n+1}) - \frac{p^{n+1} - p^{n}}{\Delta t} + S_h^{n+1}$$

In the above equations, the operators H, G denote the finite difference representations of the convective and diffusive fluxes of velocity U_i, static enthalpy h, respectively. The operator Δ_i I represents the first order Euler finite difference equivalent of $\partial/\partial x_i$. The source terms, S_{U_i}, S_h, contain all other terms which cannot be included in the convective and diffusive terms for each variable.

The simplest way to ensure that the solution procedure of the finite difference equations is stable without losing accuracy, is to separate the diagonal elements of the operators and to shift them to the left-hand side of the equations. Focusing on the momentum equation:

$$\mathrm{H}(\mathrm{U}_i) = \mathrm{H}'(\mathrm{U}_i) - \mathrm{A}_p U_i$$

where, H' is the operator of convection and diffusion term at neighbor points surrounding the main point P and A_p is the coefficient of the diagonal element of the operator H. Thus, a general implicit discretized equation can be written as:

$$(\frac{r^{n+1}}{\Delta t} + A_p)f_p^{n+1} = F' + \frac{(rf_p)^{n}}{\Delta t} + S_f$$

The detailed formula of A_p and F' are:

$$F' = \sum_{m=1}^{nb} A_m f_m^{n+1}$$

$$A_p = \sum_{m=1}^{nb} A_m$$

where, A_m is link coefficient. The time term of transportation equation is discretized as:

$$\frac{\partial r f}{\partial t} = \frac{r^n f^{n+1} - r^{n-1} f^n}{\Delta t}$$

Thus, all difference equations become

$$(\frac{r^n}{\Delta t} + A_p) f_p^{n+1} = F' + \frac{r^{n-1} f^n}{\Delta t} + S_f$$

For momentum equations, we will split the finite difference equation into several steps.

Predictor Step

The momentum equation is solved implicitly in the predictor step. Let superscript n denote previous time step. The predictor step of the momentum equation is:

$$(\frac{r^n}{\Delta t} + A_p) U_i^* = H'(U_i^*) - \Delta_i p^n + \frac{r^{n-1} U_i^u}{\Delta t} + S_{U_i}$$

In solving this equation, the new velocity U_i^* will be obtained. However, U_i^* does not satisfy the continuity equation in general since the pressure values are not correct at this step. Therefore it is necessary to construct corrector steps so that the velocity satisfies the mass conservation.

First Corrector Step

After obtaining U_i^*, the temperature T^* is calculated from static enthalpy:

$$h^n = h(T^*)$$

The intermediate density r^{nT} on p^n and T^* is defined as,

$$r^{nT} = \frac{p^n}{RT^*}$$

If we can obtain the correct values of the pressure p^*, it is possible to solve the momentum equation to obtain the new velocity U_i^{**} I satisfying the continuity equation. Then the difference equation form of the momentum equation will be

$$(\frac{p^n}{\Delta t} + A_p)U_i^{**} = H'(U_i^*) - \Delta_i p^* + \frac{r^{n-1}U_i^n}{\Delta t} + S_{U_i}$$

It should be mentioned that H' is kept unchanged at the first corrector step. The continuity equation will take the form:

$$\frac{1}{\Delta t}(r^* - r^n) + \Delta_i(r^* U_i^{**}) = 0$$

where,

$$r^* = \frac{p^*}{RT^*}$$

The values of r^* and U_i^{**} I are unknown, as we do not know p^*. The equation (3.12) is then approximated as:

$$\frac{1}{\Delta t}(r^* - r^n) + \Delta_i(r^{nT}U_i^{**}) + \Delta_i[(r^* - r^{nT})U_i^*] = 0$$

The first term can be split into,

$$\frac{1}{\Delta t}(r^* - r^n) = \frac{1}{\Delta t}(r^* - r^{nT} + r^{nT} - r^n)$$

Subtracting the predictor step momentum equation from the difference form of the momentum equation, we get

$$(\frac{r^n}{\Delta t} + A_p)(U_i^{**} - U_i^*) = -\Delta_i(p^* - p^n)$$

Defining D_u as

$$D_u = (\frac{r^n}{\Delta t} + A_p)$$

The equation becomes

$$U_i^{**} = U_i^* - D_u[\Delta_i(p^* - p^n)]$$

Substituting equations, the first pressure correction equation is obtained:

$$[\frac{1}{\Delta t R T^*} + \Delta_i(\frac{U_i^*}{RT^*}) - \Delta_i(r^{nT} D_u \Delta_i)](p^* - p^a) = -[\frac{r^{aT} - r^n}{\Delta t} + \Delta_i(r^{aT} U_i^*)]$$

Solving this equation for the pressure p^*, the new density r^*, and U_i^{**} can be obtained. Following this, the energy equation is solved implicitly, with unsteady pressure term expressed as

$$\frac{\partial p}{\partial t} = \frac{p^* - p^n}{\Delta t}$$

to obtain the new static enthalpy h^*. The new temperature T^{**} can be easily obtained from the relation between temperature are enthalpy.

Second Corrector Step

Since the effects of velocities at the neighboring points are neglected at the first corrector step, they must be accounted for at the second corrector step. Another intermediate density r^{*T} is defined as

$$r^{*T} = \frac{p^*}{RT^{**}}$$

Let p^{**}, p^{**}I and ρ** be the newest pressure, velocity and density respectively which must be solved for at this step. The momentum equation, now, is

$$(\frac{r^n}{\Delta t} + A_p)U_i^{***} = H'(U_i^{**}) - \Delta_i p^{**} + \frac{r^{n-1} U_i^n}{\Delta t} + S_{U_i}$$

Subtracting the difference equation form of the momentum equation, we get

$$(\frac{r^{n}}{\Delta t}+A_{p})(U_{i}^{***}-U_{i}^{**})=H^{'}(U_{i}^{**}-U_{i}^{*})-\Delta_{i}(p^{**}-p^{*})$$

Using the notation for D_u, we have

$$U_{i}^{***}=U_{i}^{**}+D_{u}[H^{'}(U_{i}^{**}-U_{i}^{*})-\Delta_{i}(p^{**}-p^{*})]$$

The continuity equation takes the approximated form

$$\frac{1}{\Delta t}(r^{**}-r^{n})+\Delta_{i}(r^{*T}U_{i}^{***})+\Delta_{i}[(r^{**}-r^{*T})U_{i}^{**}]=0$$

where,

$$r^{**}=\frac{p^{**}}{RT^{**}}$$

Substituting equations, and splitting the first term of the continuity equation as

$$\frac{1}{\Delta t}(r^{**}-r^{n})=\frac{1}{\Delta t}(r^{**}-r^{*T}+r^{*T}-r^{n})$$

The correction equation for the newest pressure p^{**} becomes:

$$[\frac{1}{\Delta tRT^{**}}+\Delta_{i}(\frac{U_{i}^{**}}{RT^{**}})-\Delta_{i}[r^{*T}D_{u}\Delta_{i})](p^{**}-p^{*})$$
$$=-[\frac{r^{*T-}r^{n}}{\Delta t}+\Delta_{i}(r^{*T}U_{i}^{**})]-\Delta_{i}[r^{*T}D_{u}H^{'}(U_{i}^{**}-U_{i}^{*})]$$

Solving the equation above to obtain the newest pressure p^{**}, the newest velocity U_{i}^{***} can be obtained. After this correction step, U_{i}^{***} satisfies mass conservation. From the work of [l], it can be proved that the errors introduced by the operator-splitting procedure is less than the truncation errors of the finite difference scheme used in the governing equations.

The resulting velocity and pressure were assigned to the new time level:

$$p^{n+1} = p^{**}$$

$$U_i^{n+1} = U_i^{***}$$

The relationship between total enthalpy and temperature is, now

$$T^{n+1} = T^{***}$$

$$r^{n+1} = \frac{p^{**}}{RT^{***}}$$

References

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